9.3 Basic definitions and properties of representations

Let us now return to representations. As I mentioned earlier *groups* encode abstract symmetries but *representations* describe concrete realisations of those symmetries. Informally, a representation of a group captures the action of a group on a vector space (e.g. on quantum states). In particular, in a quantum context, it is a map from the elements of a group to a set of unitaries such that multiplication of that set of unitaries obeys the same properties as the original group. For example, the group Z_2 can be represented as $\{1, X\}$ and $\{1, \text{SWAP}\}$ acting on C^2 and $(C^2)^{\otimes 2}$ respectively. We can formally define the notion of a representation of a group via the notion of homomorphisms introduced above.

Definition 9.3.1 (Group representation)**.** A representation R of a group G on a vector space V is a group homomorphism [5] from G to a set of matrices that act on a vector space V. The dimension of a representation R is defined to be the dimension of the vector state V, i.e., $\dim(R) = \dim(V)$.

We can think of this pictorially as:

$$g_1 \cdot g_2 = g_1 \cdot g_2$$

$$\downarrow \quad \downarrow \quad \downarrow \quad \downarrow$$

$$R(g_1) \cdot R(g_2) = R(g_1 \cdot g_2)$$

where $R(g)$ is a $d \times d$ dimensional matrix that acts on a d dimensional vector space V.

We stress that formally a representation is by definition the *map* R. However, more informally the word representation is used in multiple ways. For example, informally you might hear someone discuss the $\{1, \text{SWAP}\}$ representation of $\mathbb{Z}_2$. Technically $\{1, \text{SWAP}\}$ is a group (that is isomorphic to $\mathbb{Z}_2$) and the representation is the map R such that $R(e) = I$ and $R(a) = \text{SWAP}$ (where the properties of a and e are captured by the $\mathbb{Z}_2$ Cayley table). As long as you remember that fundamentally it is the underlying map that is the representation, this casual way of speaking shouldn't cause too much confusion in practise[6].

Let us give a few examples:

Trivial representation. All groups admit a trivial representation (or the Identity representation): $\forall g \in G, R(g) = I$.

Examples representations for the parity group $Z_2 = \{e, a\}$.

- As we said before we have the representations $G = \{1, X\}$ and $G_{\text{SWAP}} = \{1, \text{SWAP}\}$ acting on C^2 and $(C^2)^{\otimes 2}$ respectively. You could also have[7] $G = \{1, Z\}$ on C^2.
- On $\mathbb{R}$ it has two representations: 1) the trivial representation $R(g) = 1$ for $g = e, a$, as well as 2) the representation $R(e) = 1, R(a) = -1$.

[5]in most cases we will look at it will also be an isomorphism, i.e., a one-to-one map

[6]This subtlety is put nicely in Representation Theory for Geometric Quantum Machine Learning: *As an unfortunate feature of the subject, the word "representation" can equivalently refer to the group homomorphism R, the vector space upon which it acts V, or the image subgroup $R(G) \subset GL(V)$. Once one gets used to this, it is not as bad as it sounds: in practice, one often thinks of a representation as being the shared data of the vector space V and the linear action of G on that vector space.*

[7]This is in fact *equivalent* to the $G = \{1, X\}$ as they are related by a unitary transformation. More on equivalent transformations in a bit.

- The trivial representation $\{I\}$ can also of course be defined on a vector space of any dimension.

Examples representations for $O(3)$. Consider $O(3)$ the group of orthogonal matrices in dimension $d = 3$. We recall that this is the set of all 3×3 matrices M such that $MM^T = I$.

- The simplest representation, called the fundamental representation, is simply the set of all 3×3 orthogonal matrices.
- The morphism $R(g) = \det(M) = \pm 1$ is a representation of $O(3)$ on the vector space $\mathbb{R}$ (indeed $\det(AB) = \det(A)\det(B)$).

Fundamental representation of continuous groups. All continuous groups have the a 'fundamental' representation where the matrices in the group and the matrices in the representation coincide ("up to change of basis")[8].

Adjoint representation. Another important representation that is possible for any group is the adjoint representation. Thus far we have considered representations that map vectors to vectors, it is also possible to consider representations that map matrices to matrices. Let $V = M_2(\mathbb{C})$ denote the set of 2×2 complex matrices. The linear super-operator

$$A \mapsto U_g A U_g^\dagger \tag{9.16}$$

where $U_g = R(g)$ is a possible representation of G. For example, $U...U^\dagger$ for $U \in SU(2)$ is a representation of $SU(2)$.

So far we have spotted the representations corresponding to a symmetry group just by 'seeing them'. In fact, as I discussed earlier, the process often in physics goes the other way around. We know the symmetry at the level of the representation and then abstractify to identify the underlying group. But what about going the other way around - what if we have a group, and don't know any of its (non-trivial) representations, and want to find one?

Regular representation of finite groups. All finite groups admit what is known as the 'regular' representation as one of its representations.

Definition 9.3.2 (Regular representation)**.** For a finite group of order h, one can construct the so-called regular representation using $h \times h$ matrices as follows. First start from the following *reordered* Cayley table (here for $h = 3$):

$$C = \begin{array}{c|ccc} * & e & a^{-1} & b^{-1} \\ \hline e & e & a^{-1} & b^{-1} \\ a & a & e & ab^{-1} \\ b & b & ba^{-1} & e \end{array} \tag{9.17}$$

Now the representation can be done using the following matrices for $g \in G$: We use a matrix which is zero everywhere except for the position that corresponds to the group element in the Cayley table:

$$(R(g))_{ij} = \delta_{g,C_{ij}} \tag{9.18}$$

[8]Note that although the matrices between the group G and its representatives $\{R_g : g \in G\} \subseteq \mathrm{GL}(V)$ are identical, we think of the abstract group and its representatives as conceptually distinct.

With this definition, e is represented by the identity matrix $R(e) = I$. It is easy to check that these matrices indeed follow the group algebra. You'll work through some examples of this in the problem sheet.

It is also possible to construct representations from a simpler (set of) already known representations.

Equivalent representations. Consider a group G and a representation $R(g) \forall g \in G$. We define now $R'(g) = SR(g)S^{-1}$ where S can be any invertible matrix (in practise, in most cases we come accross, it will be a unitary matrix). This is a *similarity* transformation[9]. It is easy to see that similarity transformations of representations are still representations. It is straightforward to verify that $R'(g)$ is a representation of G (i.e., if $R(gh) = R(g)R(h)$ then $R'(gh) = SR(g)R(h)S^{-1} = SR(g)S^{-1}SR(h)S^{-1} = R'(g)R'(h)$).

Definition 9.3.3 (Equivalent representation)**.** Two representations R and R' are equivalent if they are related by a similarity transformation $R'(g) = SR(g)S^{-1}$.

Roughly speaking, representations are equivalent if we can transform one to the other by a linear invertible transformation. If what follows, we shall be mainly concerned by unitary representations and transformations. In this case $SS^\dagger = 1$ and $S^\dagger = S^{-1}$. This means that we shall consider two representations as equivalent if they simply correspond to a change of basis: $R'(g) = UR(g)U^\dagger$.

Tensor product representation. For example, consider two representations R_1 and R_2 for a group G, it is straightforward to verify (*check this!*) that the tensor product of their representations $R_1 \otimes R_2$, i.e. the set of matrices such that

$$R_1(g) \otimes R_2(g) \tag{9.19}$$

for each element g, is also a representation. For example, $\{I \otimes I, Z \otimes Z\}$ is a representation of Z_2 (in fact, $\{I^{\otimes k}, Z^{\otimes k}\}$ is a representation for any k).

Tensor product representations are fundamental in physics whenever we take the symmetry property of a single system and want to study the properties of a composite system. For example, suppose we have a system of n particles each of which are $SU(2)$ symmetric. In this case, we will be interested in the representation of $SU(2)$ on $(C^2)^{\otimes n}$, and so a natural choice is $SU(2)^{\otimes n}$.

Direct sum representation. Another useful composite representation, one that plays a key role in physics, is the direct sum representation.

Definition 9.3.4. Consider two representations R_1, R_2 of a group G acting on vector space V_1, V_2. The direct sum $R_1 \oplus R_2$ is a representation of G acting on $V_1 \oplus V_2$ defined by

$$(R_1 \oplus R_2)(g)(v_1, v_2) := (R_1(g)v_1, R_2(g)v_2), \quad \text{for all } g \in G. \tag{9.20}$$

Or, writing the matrices out explicitly, $R_1 \oplus R_2$ acting on $V_1 \oplus V_2$ we have:

$$(R_1 \oplus R_2)(g) := \begin{pmatrix} R_1(g) & 0 \\ 0 & R_2(g) \end{pmatrix}, \quad \text{for all } g \in G. \tag{9.21}$$

[9]In linear algebra, two $n \times n$ matrices A and B are called similar if there exists an invertible n-by-n matrix P such that $B = P^{-1}AP$.

That this is indeed a representation follows straightforwardly from the block structure of Eq. (9.21). (If this isn't immediately clear to you, do work through it explicitly). We can also take the direct sum of the same representation, i.e., $R_1 \oplus R_1$, in which case we say that R_1 has multiplicity of two, and we write

$$(R_1 \oplus R_1)(g) = \begin{pmatrix} R_1(g) & 0 \\ 0 & R_1(g) \end{pmatrix} = I \otimes R_1(g), \quad \text{for all } g \in G. \tag{9.22}$$

Notice that due to the block structure of a direct sum representation the action of an element of the representation structure of a group leave certain subspaces invariant. This will turn out to be very important.

Hopefully it is now clear how you can take simple representations of a group and create more complex ones. In many cases, we will in fact be more interested in going in the other direction. Taking a complex representation and trying to break it down into a simpler one. More concretely, one of the things representation theory is most useful for is *taking a representation (e.g. say a tensor one), and expressing it as a direct sum of representations on smaller subspaces.* We will discuss this in Section 9.4

9.4 (Ir)Reducible Representations of Groups

Our goal here will be discuss when/how it is possible to decompose a representation into a direct sum of other representations and, hopefully, give a sense of why we might be interested in doing this in the first place.

9.4.1 Warm up example

Consider a two qubit system and the tensor product representation of $SU(2)$ on this space, i.e.

$$R(g) = U_g \otimes U_g\,. \tag{9.23}$$

Can we decompose this into the direct sum of two other representations? That is, can we block diagonalize $U_g \otimes U_g$, i.e., write it in the form

$$U_g \otimes U_g = \begin{pmatrix} A(g) & 0 \\ 0 & B(g) \end{pmatrix} \tag{9.24}$$

for some matrices A and B for all g

To answer this we first note that $U_g \otimes U_g$ commutes with the SWAP operator $[U_g \otimes U_g, \text{SWAP}] = 0$. This means that it is possible to (block[10]) diagonalize $U_g \otimes U_g$ in the same basis as the SWAP. More generally, the following proposition holds.

Proposition 9.4.1. *Let $R(g) = U_g$ be a representation of a group G, and let H be a Hermitian operator such that $[U_g, H] = 0$ for all $g \in G$. Then, for any eigenvector $|\psi\rangle$ of H with eigenvalue λ, $U_g|\psi\rangle$ is also an eigenvector of H of eigenvalue λ. That is, H is simultaneously block diagonalized with U_g.*

[10] The fact we have 'block diagonalized' rather than simply 'diagonalized' here allows for the fact that H and U_g can have degenerate eigenvalues

Demo. Observe that $HU_g|\psi\rangle = U_g H|\psi\rangle = \lambda U_g|\psi\rangle$. This means that H and U_g are (block) diagonal in the same basis [11]. □

Next we recall that the SWAP operator has eigenvalue 1 on the symmetric subspace spanned by the degenerate eigenstates $\{|11\rangle, |01\rangle + |10\rangle, |00\rangle\}$ and eigenvalue -1 on the anti-symmetric subspace spanned by $\{|10\rangle - |01\rangle\}$. That is, it is block diagonalized in the symmetric-antisymmetric decomposition.

It thus follows that the tensor representation $U_g \otimes U_g$ is also block diagonalized by the symmetric-antisymmetric decomposition of V: i.e., in the basis $\{|11\rangle, \frac{1}{\sqrt{2}}(|01\rangle + |10\rangle), |00\rangle, \frac{1}{\sqrt{2}}(|10\rangle - |01\rangle\}\}$. That is, every representative $U_g \otimes U_g$ can be expressed as

$$U_g \otimes U_g = \left(\begin{array}{ccc|c} & & & 0 \\ & & & 0 \\ & & & 0 \\ \hline 0 & 0 & 0 & \end{array}\right) \tag{9.25}$$

where □ indicates the blocks to be filled in with the appropriate matrix elements. That is, the claim is that if you take any matrix constructed from the tensor product of two single qubit matrices and write it in the Bell basis [12], it will have the block diagonal form shown above [13].

Note that this decomposition, Eq. (9.25), also implies the existence of *invariant subspaces* under the action of $U_g \otimes U_g$. Concretely, we get straight away that the state $|\Phi^-\rangle = \frac{1}{\sqrt{2}}(|01\rangle - |10\rangle)$ is left unchanged by any $U_g \otimes U_g$ where $U_g \in SU(2)$. This is pretty cool, right? And would not have been obvious without group theory. Similarly, any state that lives in the span $\{|11\rangle, |01\rangle + |10\rangle, |00\rangle\}$ will remain in that subspace.

More formally, using the notation $\mathrm{Sym}^2(\mathbb{C}^2)$ for the symmetric subspace and $\mathrm{Alt}^2(\mathbb{C}^2)$ for the antisymmetric subspace, we can write the composite vector space as $V = \mathrm{Sym}^2(\mathbb{C}^2) \oplus \mathrm{Alt}^2(\mathbb{C}^2)$ and it is possible to construct representations that act on these spaces separately. Concretely, it can be built from the direct sum of $SU(1)$ (i.e. just the 1 by 1 identity matrix) on the subspace $\mathrm{Alt}^2(\mathbb{C}^2)$ and $SU(3)$ on the $\mathrm{Sym}^2(\mathbb{C}^2)$ subspace. Note also, that due to the block structure of $U_g \otimes U_g$ a state in the subspace $\mathrm{Sym}^2(\mathbb{C}^2)$ remains in the subspace spanned by $\mathrm{Sym}^2(\mathbb{C}^2)$ (and similarly for $\mathrm{Alt}^2(\mathbb{C}^2)$). Again, if this feels a bit abstract - check it numerically!

It is important to stress that it is not always possible to reduce a representation into a direct sum of representations. Or, equivalently, a representation will not always have an invariant subspace. For a simple example of such an *irreducible* representation consider the fundamental representation of $SU(2)$. This is simply the continuous set of all single qubit unitaries. Clearly there is no single basis in which such matrices are all diagonal. Or, equivalently, there is no way to split the vector space into disjoint subspaces where any vector in that space remains in that space under any arbitrary single qubit unitary. Similarly, the representation

[11] Note the similarity with our discussion in Eq. (9.3)- the calculation was exactly the same but there we used it to argue that symmetries indicated degeneracies. We will come back to this perspective again in a bit when we see that the dimension of irreducible representations indicates the number of degenerate eigenstates. This probably won't make much sense now if you're reading these notes through for the first time, but hopefully this is helpful if reading back through.

[12] The subspace spanned by $\{|11\rangle, |01\rangle + |10\rangle, |00\rangle\}$ is alternatively spanned by the Bell states $\{|\Phi^+\rangle, |\Phi^-\rangle, |\Psi^+\rangle\}$.

[13] *Exercise:* If you're not yet fully convinced, check this numerically. It's quite cool to see it work in practise. I've uploaded a mathematica file to the moodle where I run through it. You can get a free mathematica licence from EPFL. That said, you could also quickly check this in python / whatever your favourite language is.

$SU(2)$ on $\mathrm{Sym}^2(\mathbb{C}^2)$ and $\mathrm{Alt}^2(\mathbb{C}^2)$ cannot be further reduced (e.g. there is no subspace within $\{|11\rangle, |01\rangle + |10\rangle, |00\rangle, |10\rangle - |01\rangle\}$ that remains invariant under any unitary $U \otimes U$ with $U \in SU(2)$).

Before we move on to discussing when representations are and are not reducible let me just highlight that there is lots of physics in the simple example of decomposing $SU(2) \otimes SU(2)$ into a direct sum. And this physics hopefully gives you a sense of why reducing representations is physically interesting.

Link with identical particles. Firstly, thinking back to when we studied identical particles, you should recognise the symmetric and anti-symmetric subspaces found above as corresponding to Bosons and Fermions respectively. Thus these observations could be seen as another way of showing[14] that there are two types of fundamental particles that we cannot transform between.

Link with addition of angular momentum/Clebsch-Gordan coefficients. The two blocks found above also correspond to the spin 1 and spin 0 blocks obtained when adding the momentum of two spin half particles. That is, we have three spin 1 states:

$$|s = 1, m = 1\rangle = |11\rangle \tag{9.26}$$

$$|s = 1, m = 0\rangle = \frac{1}{\sqrt{2}}(|10\rangle + |01\rangle) \tag{9.27}$$

$$|s = 1, m = -1\rangle = |00\rangle \tag{9.28}$$

and one spin 0 state:

$$|s = 0, m = 0\rangle = \frac{1}{\sqrt{2}}(|10\rangle - |01\rangle)\,. \tag{9.29}$$

Here the left hand side of the equations denotes the state corresponding to the total spin $s = s_1 + s_2$ of two spin 1/2 particles ($s_1 = 1/2$, $s_2 = 1/2$) and total spin m orientated in the z direction. On the right hand side of the equations we denote the spin orientation of the two particles, e.g. $|10\rangle$ corresponds to one spin aligned spin up with z and the other spin pointing down in the z direction. These equations, read right to left, can be viewed as representing a change in basis from a basis where we list the individual orientations of the spins to the resulting total spin and orientation of the combined spins. Thus we see that the decomposition of a tensor product representation into a direct sum of representation has a deep link with how to add the angular momentum of composite systems. We'll come back to this in a lot more detail in a couple of weeks times.

9.4.2 Definitions of (Ir)Reducibility.

Hopefully that example gave you some hint of what we mean by *reducing* representation into a direct sum of representations. Hopefully it also gave you a hint as to why it is physically interesting. I appreciate right now it might seem like an overkill and all we have done is rephrase ideas from quantum physics 1 in a group theoretic language. However, in more complex scenarios we will start only with the symmetry properties and be faced with the challenge of trying to identify the relevant subspaces. This is when group and representation theory really becomes useful.

[14] Technically we just consider the rather trivial $U \otimes U$ evolutions here but the more general set of evolutions that commute with SWAP could similarly be diagonalized in the symmetric and anti-symmetric subspaces.

Let's define the concepts of reducible and irreducible representations a little more formally.

Definition 9.4.2 (Reducible representation)**.** A representation $R(g)$ of a group G over a vector space V is reducible if there exists an invariant subspace. That is, if there exists a non-trivial (i.e. not just V or $\mathbf{0}$) subspace W of V such that $\forall |w\rangle \in W$, we have $R(g)|w\rangle \in W$, for any element $g \in G$.

In plain words: an invariant subspace means a smaller space than the actual space V, where the application of any matrix in the representation does not leave the space. In terms of matrices, this means that there is an equivalent representation that can be written as a block matrix with a zero block:

$$R(g) = \begin{pmatrix} Q(g) & 0 \\ T(g) & P(g) \end{pmatrix} \tag{9.30}$$

In fact if we write all vectors in V as $|x\rangle = \begin{pmatrix} v \\ w \end{pmatrix}$, we see that the subspace defined by vectors $|w\rangle = \begin{pmatrix} 0 \\ w \end{pmatrix}$ is transformed as

$$R(g)|w\rangle = \begin{pmatrix} 0 \\ P(g)w \end{pmatrix} \tag{9.31}$$

so that such vectors never leave the subspace. If a representation is reducible, then there is a basis such that all matrices can be written as such block matrices in the basis.

Definition 9.4.3 (Irreducible representation)**.** An irreducible representation is a representation that is not reducible.

Obviously, representations that live in dimension 1 are irreducible. One of the main uses of group theory in quantum mechanics is to *reduce* representations into a set of irreducible ones.

A particular case of reducibility is *complete reducibility*, in which case $T(g) = 0$ as well.

Definition 9.4.4 (Completely Reducible representation)**.** A representation $R(g)$ of a group G is completely reducible if it splits into a direct sum of irreducible representations

$$R(g) = \begin{pmatrix} R_1(g) & 0 & \dots & 0 \\ 0 & R_2(g) & \dots & 0 \\ \vdots & & & \\ 0 & 0 & \dots & R_k(g) \end{pmatrix} = \bigoplus_i R_i(g)\,. \tag{9.32}$$

We may wonder if all reducible transformations are completely reducible. Sadly, this is not the case. Here is an example: the matrices

$$M(x) = \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \tag{9.33}$$

are a representation of the group $\mathbb{R}, +$. Indeed, $M(x)M(y) = M(x+y)$. However, we cannot diagonalize such matrices.

The good news, however, is that in this lecture we will limit ourselves to *unitary representations* which if they are reducible are always completely reducible[15].

[15] To see this note that since unitary transformation send orthogonal states to orthogonal states $T(g)$ must be zero in equation (9.30).

Much of the rest of this chapter will be centred around developing the tools to find and use irreducible representations. More precisely, we are going to do two things: i) Study the consequences of having an irreducible representation, and ii) See how to get an irreducible representation. Irreducible representations are often called 'irreps' for short.

A word of warning, the next few sections will get pretty technical. This is unavoidable. If you are ever lost, try and construct yourself some examples of the statements being made. To avoid getting too bogged down in technicalities many of the longer proofs will be left to appendices/references. These proofs are non-examinable - but you may find working through them helpful for your understanding.

9.5 Schur's Lemmas

A key result to help identify irreps is Schur's lemma. This discusses the link between irreducible representations, and in particular their link with an operator that commutes with all elements of the representation.

Schur's first lemma gives us a criterion to determine when two representations are reducible.

Lemme 9.5.1 (Schur's First Lemma[16]). *Let $R_1(g)$ and $R_2(g)$ be two non-equivalent irreducible representations of a group G, each acting on vector spaces V_1 and V_2. If there is a matrix A is such that*

$$AR_1(g) = R_2(g)A \quad \forall g \in G \tag{9.34}$$

then $A = 0$.

Or, turning it around, if you can find an A that satisfies Eq. (9.34) such that $A \neq 0$ then you representations R_1 and R_2 are reducible. This therefore gives you one way of detecting that a representation is reducible.

The second lemma studies what kind of matrices commute with all matrices of a given irreducible representation.

Lemme 9.5.2 (Schur's Second Lemma[17]). *Let R be an irreducible unitary representation [18] of a group G. If*

$$AR(g) = R(g)A \quad \forall g \in G,$$

then $A = \lambda I$ for some $\lambda \in \mathbb{C}$.

I quite like the explanation of Schur's second lemma in Group Theory In A Nutshell For Physicists so I'll quote from it directly: *If I give you a bunch of matrices $R_1, R_2, \cdots, R_n$, the identity matrix I commutes with all these matrices, of course. But it is also quite possible for you to find a matrix A, not the identity, that commutes with all n matrices. The theorem says that you can't do this if the given matrices $D_1, D_2, \cdots, D_n$ are not any old bunch of matrices you found*

[16] The proof here isn't too bad.

[17] For a nice proof of this check out Group theory in a nutshell for physicists.

[18] For those of you for which these details are important (and/or those who have been confused how Schur's lemma is stated differently in different books/references) the statement and proof of Schur's Second Lemma can differ slightly depending on whether you are looking at finite or infinite dimensional representations. However, we will not worry about these subtleties in this course. It holds in the form stated here for finite or compact unitary representations (i.e. all representations we will be interested in for this course).

hanging around the street corner, but the much-honored representation matrices furnishing an irreducible representation of a group.

In short, if there exists an operator A that commutes with all elements of two *irreducible* representations then Schur lemmas gives a very strong limit to what A can be: either a trivial diagonal matrix, if the representations are equivalent (i.e., the same up to a change of basis), or a zero one, if they are not. Or, turning it around, no operator - except the trivial zero operator - commutes with all elements of two non-equivalent irreducible representations. So if you find a non-trivial operator that does commute then the representations are reducible.

Example. To make this less abstract let's first consider our favourite example of $SU(2) \otimes SU(2)$. We know that its irreps are $SU(1)$ on the anti-symmetric subspace $\text{Alt}^2(\mathbb{C}^2)$ and $SU(3)$ on the symmetric subspace $\text{Sym}^2(\mathbb{C}^2)$. It follows from Schur's Second Lemma that the only operators that commute with $SU_3(g)$ *for all* g is a scalar multiplication of I on this subspace, i.e. $I = |\Psi^+\rangle\langle\Psi^+| + |\Phi^-\rangle\langle\Phi^-| + |\Phi^+\rangle\langle\Phi^+|$. And this is, of course, indeed the case.

As another example of how to apply Schur's lemma let us consider the $R(e) = I$ and $R(a) = X$ representation of Z_2 group. The $A = X \neq I$ operator commutes with both I and X and so we know immediately that $R(e) = I$ and $R(a) = X$ is not an irrep. Note, that this is a consequence of the Z_2 group being Abelian. More generally, from Schur's lemma, we can deduce something very important:

Theorem 9.5.3 (Representation of Abelian groups)**.** *All irreducible representations of Abelian groups are scalar.*

Demo. Let $R(g)$ be an irreducible representation of an Abelian group G. Then we have, $\forall g, h \in G$, $R(g)R(h) = R(g * h) = R(h * g) = R(h)R(g)$. Since $R(h)$ commutes with all $R(g)$, then from the second Schur lemma, it must be a matrix $I\lambda$, and $R(h) = I\lambda(h)$ for all h. Since it is also irreducible, then $R(h) = \lambda(h)$ (i.e. $= I\lambda$ clearly has invariant subspaces for $\dim(I) \geq 2$). □

More generally, given a bunch of matrices, there are potentially many matrices that commute with all of them. However, if the matrices form an irreducible representation of a finite group only multiples of the identity matrix commute with them. In general, we will be interested in problems where the Hamiltonian commutes with a given symmetry of a system (and so are block diagonal in the same basis). This means that if we can identify the systems irreps we can block diagonalize the Hamiltonian. Let's go through this argument more carefully.

9.6 Irreps are all about Block Diagonalization!

In a quantum context one often considers the Hamiltonian H, and G a symmetry group that commutes with H. More precisely, suppose we have a representation of a symmetry group over a Hilbert space $\mathcal{H}$ with $[R(g), H] = 0 \ \forall \ g \in G$. For example, $\mathcal{H}$ could be an infinite dimensional space, that forms a basis (for instance the Fourier basis). In an infinite dimensional space, we expect that $R(g)$ is reducible. So, if we work hard, we can find a basis of the Hilbert space that reduces the representation, that is we can recompose the space as $\mathcal{H} = \mathcal{H}_1 \oplus \mathcal{H}_2 \oplus \dots$ where all the $\mathcal{H}_i$ are invariant over the group transformation. At this point, we thus have $\forall g \in G, R(g) = R_1(g) \oplus R_2(g) \oplus R_3(g) \dots$ where each of the R_i are irreps, or equivalently in

Figure 9.4:

matrix form:

$$R(g) = \begin{pmatrix} R_1(g) & 0 & 0 & \dots \\ 0 & R_2(g) & 0 & \dots \\ 0 & 0 & R_3(g) & \dots \\ \dots \end{pmatrix}. \tag{9.35}$$

In this basis, we write the Hamiltonian (which is of course Hermitian) as

$$H = \begin{pmatrix} H_{11} & H_{12} & H_{13} & \dots \\ H_{21} & H_{22} & H_{23} & \dots \\ H_{31} & H_{32} & H_{33} & \dots \\ \dots \end{pmatrix} = \begin{pmatrix} H_{11} & H_{12} & H_{13} & \dots \\ H_{12}^* & H_{22} & H_{23} & \dots \\ H_{13}^* & H_{23}^* & H_{33} & \dots \\ \dots \end{pmatrix} \tag{9.36}$$

Now, let us see what Schur's lemma tells us. If $[R(g), H] = 0 \ \forall \ g \in G$ then we can apply the Schur lemma between all blocks in this decomposition. Writing out the matrices explicitly, and using $R(g)H = HR(g)$, we see that on the diagonal we have

$$H_{kk}R_k = R_k H_{kk} \tag{9.37}$$

for all k and so by Schur's second lemma along the diagonal we have $\lambda_k I$. Then on the off-diagonal we have terms of the form

$$H_{jk}R_k = R_j H_{jk}\,. \tag{9.38}$$

If R_k and R_j are non-equivalent then, from Schur's first lemma, the block $H_{jk} = 0$. If R_k and R_j are equivalent then the block H_{jk} can be non-zero. That is, assuming only R_1 and R_2 are equivalent, the Hamiltonian can be written as

$$H = \begin{pmatrix} \lambda_1 I & H_{12} & 0 & 0 & \dots \\ H_{21} & \lambda_2 I & 0 & \dots \\ 0 & 0 & \lambda_3 I & 0 & \dots \\ 0 & 0 & 0 & 0 & \dots \\ \dots \end{pmatrix} \tag{9.39}$$

This allows us to considerably simplify the Hamiltonian just from the role of symmetry. In fact, if all the representations are non-equivalent then all the off diagonal terms will have vanished and we have block diagonalized the Hamiltonian - i.e. we know the degenerate eigenspaces of the Hamiltonian! This then makes finding the eigenvalues/eigenvectors of a Hamiltonian much easier as we can just find the eigenvalues/vectors of the individual blocks (which are smaller and so easier to handle!) rather than work with the large composite Hamiltonian.

Note, that if all the representations are non-equivalent then we can immediately read off the degeneracy of each of each of the eigenvalues- it's just given by the dimension of the irreps! Thus, as promised at the end of the warm up at the start of the group theory lectures, e.g. after Eq. (9.3), group and rep theory allows us to not only better understand but also explicitly compute the number of degenerate eigenvalues a Hamiltonian has.

Or, turning it around, given experimental information on the degeneracy, we can use this information to try and identify the relevant symmetry group. In particular, G has to have at least one d-dimensional irreducible representation.

In summary, in quantum mechanics:

$G \Longrightarrow$ degeneracy and $G \Longleftarrow$ degeneracy
d = degrees of degeneracy = dimension of irreducible representation

Example 1: Indistinguishable particles. Lets suppose we are interested in studying a Hamiltonian H of two indistinguishable particles. The relevant symmetry group in this case is the permutation group for two objects, $S_2 = \{e, p\}$ with the cayley table:

	e	p
e	e	p
p	p	e

.

Let's say our two particles are two qubits (because qubits are nice and simple). Then a representation of this group is $R(e) = I, R(p) = \text{SWAP}_{12}$ where $\text{SWAP}_{12}|00\rangle = |00\rangle$, $\text{SWAP}_{12}|01\rangle = |10\rangle$, $\text{SWAP}_{12}|10\rangle = |01\rangle$, and $\text{SWAP}_{12}|11\rangle = |11\rangle$, or in matrix form

$$\text{SWAP}_{12} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \tag{9.40}$$

We know immediately that this representation has to be reducible because S_2 is an Abelian group and the irreps of an Abelian group are 1D. The irreps of S_2 are clearly the trivial irrep $R_1(e) = 1, R_1(p) = 1$ and $R_2(e) = 1, R_2(p) = -1$. So how do we write the $R(e) = I, R(p) = \text{SWAP}_{12}$ representation in terms of these? We we've already seen this today! The SWAP operator is diagonal in the Bell basis with eigenvalues +1 and -1. That is, in the Bell basis we can write

$$\text{SWAP}_{12} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}. \tag{9.41}$$

Thus we have:

$$R(e) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} R_1(e) & 0 & 0 & 0 \\ 0 & R_1(e) & 0 & 0 \\ 0 & 0 & R_1(e) & 0 \\ 0 & 0 & 0 & R_2(e) \end{pmatrix} \tag{9.42}$$

and

$$R(p) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} = \begin{pmatrix} R_1(p) & 0 & 0 & 0 \\ 0 & R_1(p) & 0 & 0 \\ 0 & 0 & R_1(p) & 0 \\ 0 & 0 & 0 & R_2(p) \end{pmatrix} \tag{9.43}$$

Or, more compactly, $R(g) = R_1(g) \oplus R_1(g) \oplus R_1(g) \oplus R_2(g)$. We have successfully broken down our representation down into irreps!

Ok, so now we've figured out the irreps of the relevant symmetry group and representation for our particular system, what can we say about the Hamiltonian of the system. Well we know that H commutes with all representations of the symmetry group - this is by assumption - this is what it means for a physical system to have a given symmetry. Thus we have $[H, \mathrm{SWAP}_{12}] = 0$. If we now apply Schur's lemma we know that the Hamiltonian must be block diagonalisable in the Bell basis. That is, it has to be of the form:

$$H \left(\begin{array}{ccc|c} & & & 0 \\ & & & 0 \\ & & & 0 \\ \hline 0 & 0 & 0 & \end{array} \right) \tag{9.44}$$

where $\square$ indicates the blocks to be filled in with the appropriate matrix elements.

As often with these examples, the application of group theory here right now might feel like overkill. Of course we always knew that if H commuted with SWAP it had to be block-diagonalizable in the same basis. But isn't it nice to see that these rather abstract looking theorems (Schur's lemma's) lead to the same conclusions. Or, at least, I would rather you be bored reading this thinking it all makes sense and is trivial than be completely lost. I warn you, the next example is also pretty trivial. However, example 3 on Bloch's theorem (which will be covered in more detail in the problem sheet) is where things start to get more interesting.

Example 2: Parity. A parity transformation (also called parity inversion) is the flip in the sign of a spatial coordinate. In three dimensions, it refers to the simultaneous flip in the sign of all three spatial coordinates (a point reflection): $\mathbf{P} : \begin{pmatrix} x \\ y \\ z \end{pmatrix} \mapsto \begin{pmatrix} -x \\ -y \\ -z \end{pmatrix}$. A wave function can always be decomposed into an even and an odd component $\psi(\boldsymbol{x}) = \psi^+(\boldsymbol{x}) + \psi^-(\boldsymbol{x})$, and the application of the parity operator transforms it as

$$\mathbf{P}\psi(\boldsymbol{x}) = \mathbf{P}\psi^+(\boldsymbol{x}) + \mathbf{P}\psi^-(\boldsymbol{x}) = \psi^+(-\boldsymbol{x}) + \psi^-(-\boldsymbol{x}) = \psi^+(\boldsymbol{x}) - \psi^-(\boldsymbol{x}) \tag{9.45}$$

Note in particular that $\mathbf{PP} = 1$. The set of all parity transformations that can be obtained by the parity operator is thus limited to 2. The set of of these transformations forms the parity group $Z_2 = \{e, p\}$ which is the same as the permutation group on two objects (i.e, the same group we were just looking at). So we recall again that this group has only two possible irreducible representations in dimension 1 on $\mathbb{R}$: (i) $R_1(e) = 1$ and $R(p) = 1$ and (ii) $R_2(e) = 1, R_2(p) = -1$.

Consider now a problem with a Hamiltonian that commutes with any parity transformation. This will be the case for any particle with a potential such that $V(x) = V(-x)$. The Hamiltonian lives in a large (possibly infinite) Hilbert space $\mathcal{H}$. Now, we consider a basis of $\mathcal{H}$ made of even and odd functions (such as the Fourier basis): $\{\phi_1^+(x), \phi_2^+(x), \ldots, \phi_1^-(x), \phi_2^-(x), \ldots\}$.

This basis defines invariant subspaces with respect to parity, i.e. for any possible representation R of the parity group, an even (odd) basis function stays even (odd) under any application of

$R(e)$ or $R(p)$. We can write $R(e)$ and $R(p)$ in this basis as

$$R(e) = \begin{pmatrix} 1 & 0 & 0 & 0 & \dots \\ 0 & 1 & 0 & 0 & \dots \\ 0 & 0 & 1 & 0 & \dots \\ 0 & 0 & 0 & 1 & \dots \\ \dots \end{pmatrix} \text{ and } R(p) = \begin{pmatrix} 1 & 0 & 0 & 0 & \dots \\ 0 & 1 & 0 & 0 & \dots \\ \dots \\ \dots & \dots & -1 & 0 & 0 \\ \dots & \dots & 0 & -1 & 0 \\ \dots \end{pmatrix}$$

where in $R(p)$ the rows/columns with $+1$ correspond to even basis states and the rows/columns with -1 correspond to the odd basis states. That is, we have

$$R(g) = \begin{pmatrix} R_1(g) & 0 & 0 & 0 & \dots \\ 0 & R_1(g) & 0 & 0 & \dots \\ \dots \\ 0 & 0 & R_2(g) & 0 & \dots \\ 0 & 0 & 0 & R_2(g) & \dots \\ \dots \end{pmatrix}. \tag{9.46}$$

Applying the Schur lemmas, and noting that $R_1(g)$ and $R_2(g)$ are non-equivalent irreps, we now obtain that

$$H = \begin{pmatrix} H_{11} & 0 \\ 0 & H_{22} \end{pmatrix}. \tag{9.47}$$

It follows that the eigenfunctions of a Hamiltonian that commutes with the parity operator are either odd or even. That is, they have well defined parity. You of course already knew this - but hopefully it is nice to see that this can arise from your new found understanding of irreps.

Example 3: Bloch's Theorem. Let's now consider the case of a particle moving in 1D in a periodic potential $V(x)$. That is under the Hamiltonian

$$H = \frac{p^2}{2m} + V(x) \quad \text{where} \quad V(x+a) = V(x). \tag{9.48}$$

We will suppose that the particle moves on a 1-dimensional lattice consisting of N sites and periodic boundary conditions.

What is the symmetry in group in this case? Well the Hamiltonian is left unchanged by any translation U_a by a distance a, i.e., $x \to Ux = x + a$. It follows, that the symmetry group consists of $\{I, U_a, U_a^2, \cdots, U_a^{N-1}\}$. Note that given the periodic boundary conditions we have that $U_a^N = I$. Thus the symmetry group is just the familiar cyclic group $\mathbb{Z}_N$. In the problem sheet, you'll then use your understanding of the irreps of $\mathbb{Z}_N$ to determine the form of the eigenfunctions of H.

9.7 How many irreducible representations does a group have?

Let us start by presenting two theorems that can be used to deduce the number of irreps that a group has.

Lemme 9.7.1. *Burnside lemma: For a finite group of order h, there are only a finite number n of irreducible representations $a = 1, \dots, n$ of dimension l_a, and*

$$\sum_{a=1}^{n} l_a^2 = h \tag{9.49}$$

For example, the group $\mathbb{Z}_2$ is order 2 (i.e. contains two elements). It's irreducible representations are the trivial representation, $e \to 1$ and $a \to 1$, and the sign representation, $e \to 1$ and $a \to -1$. And this satisfies the Burnside lemma as $1^2 + 1^2 = 2$. (For a proof of this Theorem see Appendix 9.10.9).

Lemme 9.7.2. *Number of Irreducible Representations: For a finite group of order h, the number of (non-equivalent) irreps is equal to the number of* ***conjugacy classes****:*

$$N_r = N_c\,. \tag{9.50}$$

To understand this second theorem, which we will prove in Section 9.8, we will need to introduce the concept of a *conjugacy class*.

9.7.1 Equivalence relations and conjugacy classes.

I thought equivalence/conjugacy classes were really nicely explained in 'group theory in a nut shell for physicists' so I'm going to quote directly from there here:

"Given a group G, distinct group elements are of course not the same, but there is a sense that some group elements might be essentially the same. The notion of equivalence class makes this hunch precise.

Before giving a formal definition, let me provide some intuitive feel for what "essentially the same" might mean. Consider $SO(3)$. We feel that a rotation through 17° and a rotation through 71° are in no way essentially the same, but that, in contrast, a rotation through 17° around the z-axis and a rotation through 17° around the x-axis are essentially the same. We could simply call the x-axis the z-axis.

As another example, consider S_3. We feel that the elements (123) and (132) are equivalent, since they offer essentially the same deal; again, we simply interchange the names of object 2 and object 3. We could translate the words into equations as follows:

$$(23)^{-1}(123)(23) = (32)(12)(23)(32) = (32)(21) = (321) = (132) \tag{9.51}$$

A transformation using (23) has turned (123) and (132) into each other, as expected. Similarly, you would think that (12), (23), and (31) are essentially the same, but that they are in no way essentially the same as (123).

In a group G, two elements g and g' are said to be equivalent ($g \sim g'$) if there exists another element f such that

$$g' = f^{-1} g f$$

The transformation $g \to g'$ is like a similarity transformation in linear algebra."

Thus the equivalence relation divides the elements of group G into distinct classes which are called conjugate classes or simply classes.

Let us consider for example the order 4 cyclic group:

$$G = \begin{array}{c|cccc} * & e & a & b & c \\ \hline e & e & a & b & c \\ a & a & e & c & b \\ b & b & c & e & a \\ c & c & b & a & e \end{array} \tag{9.52}$$

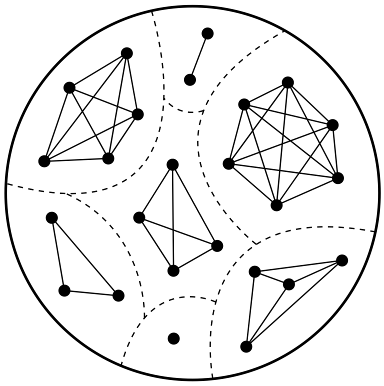

Figure 9.5: Graph of an example equivalence with 7 classes (from Wiki page on equivalence classes).) Each edge represents ~ (with edges from any node to itself not shown).

In this case, can check that we have four conjugacy classes, each containing one member. (But, for example, $\{a, b\}$ is not an equivalence class because there is no $u \in \{a, b\}$ such that $uau^{-1} = b$.)

In fact, this is true for each Abelian group (and the converse is true). An Abelian group of order n has n conjugacy classes. This is a trivial consequence of commutation (i.e. $uau^{-1} = uu^{-1}a = a = b$)! Looking back at Lemma 9.7.2 this then implies that an order n Abelian group has n irreps (irreducible representations). (Note, you could also have seen this from the fact that the irreps of Abelian groups are 1D and the Burnside Lemma).

A more interesting example is given by the S_3 (i.e., the C3v group). Here we have three conjugacy classes: $\{e\}$,$\{c_+ = (123), c_- = (132)\}$, and the three mirrors $\{\sigma = (12), \sigma' = (23), \sigma'' = (32)\}$. Note that e is always a "isolated" class in itself. Indeed, if $x = u^{-1}eu$ then $x = e$. We then already showed that (123) and (132) were equivalent in Eq. (9.51). I'll leave it as an exercise for you to convince yourself that $\sigma = (12), \sigma' = (23), \sigma'' = (32)$ are equivalent. (If you're stuck check out this video). Looking back at Lemma 9.7.2 this tells us that C3v has 3 irreps.

So we now have a way of counting how many irreps we have. This can be useful because if we are trying to find all irreducible representations of a group because it gives us a way of knowing how many we are missing. Then Burnside's Lemma gives us a way of guessing the dimensions of the missing representations. But this is only so useful. Really we want to know how to identify some of the irreps.